

Finally, the essay developes above thirty series which rise out of this theory, nearly all of which are believed to be new. The most elegant of them may find a place here. Writing, for conciseness, C so related to c that $F(c, \frac{1}{2}\pi) = \frac{1}{2}\pi C$, and $\therefore F(c, \omega) = Cx$, we have

$$(a). \omega = x + \frac{\sin 2x}{\cos \pi a} + \frac{1}{2} \cdot \frac{\sin 4x}{\cos 2\pi a} + \frac{1}{3} \cdot \frac{\sin 6x}{\cos 3\pi a} + \&c.$$

$$(b). C\Delta(c, \omega) = 1 + \frac{2 \cos 2x}{\cos \pi a} + \frac{2 \cos 4x}{\cos 2\pi a} + \frac{2 \cos 6x}{\cos 3\pi a} + \&c.,$$

$$(c). \frac{1}{8}C^2c^2 \sin 2\omega = \frac{\sin 2x}{\cos \pi a} + \frac{2 \sin 4x}{\cos 2\pi a} + \frac{3 \sin 6x}{\cos 3\pi a} + \&c..$$

$$(d). V(c, \omega) = \frac{1 - \cos 2x}{\sin \pi a} + \frac{1}{2} \cdot \frac{1 - \cos 4x}{\sin 2\pi a} + \frac{1}{3} \cdot \frac{1 - \cos 6x}{\sin 3\pi a} + \&c...$$

$$(e). \frac{1}{2}C.G(c, \omega) = \frac{\sin 2x}{\sin \pi a} + \frac{\sin 4x}{\sin 2\pi a} + \frac{\sin 6x}{\sin 3\pi a} + \&c....$$

This is virtually eq. 49 of Legendre's Second Supplement, § 7. In eq. 53 of the same, he has a development of $\sin^2 \omega$, which is given by Mr. Newman in a notation similar to eq. (c) above.

$$(f). -\frac{1}{2} \log \Delta(c, \omega) = \frac{1 - \cos 2x}{\sin \pi a} + \frac{1}{3} \cdot \frac{1 - \cos 6x}{\sin 3\pi a} + \frac{1}{5} \cdot \frac{1 - \cos 10x}{\sin 5\pi a} + \&c...$$

$$(g). \frac{1}{2}Cc \sin \omega = \frac{\sin x}{\sin \frac{1}{2}\pi a} + \frac{\sin 3x}{\sin \frac{3}{2}\pi a} + \frac{\sin 5x}{\sin \frac{5}{2}\pi a} + \&c....$$

$$(h). \frac{1}{2}Cc \cos \omega = \frac{\cos x}{\cos \frac{1}{2}\pi a} + \frac{\cos 3x}{\cos \frac{3}{2}\pi a} + \frac{\cos 5x}{\cos \frac{5}{2}\pi a} + \&c....$$

$$(i). \frac{1}{4}Cc \sqrt{\frac{1-\Delta}{1+\Delta}} = \frac{\sin x}{\sin \pi a} + \frac{\sin 3x}{\sin 3\pi a} + \frac{\sin 5x}{\sin 5\pi a} + \&c....$$

Moreover, Jacobi's two celebrated theorems follow as a corollary from the general propositions here established.

II. "On the Comparison of Hyperbolic Arcs." By C. W. MERRIFIELD, Esq. Communicated by the Rev. Dr. BOOTH.
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(Abstract.)

If in common trigonometry we take one arc equal to the sum of two others, the cosine of the first arc is equal to the product of the cosines, diminished by the product of the sines of the other two.

If we pass from the circle to the ellipse, the addition of the arcs becomes more complicated, the product of the sines being multiplied by a radical. This relation was first obtained by Euler as the integral of a differential equation. I have called it elsewhere, for convenience, *the elliptic equation* between three amplitudes. Moreover, we are no longer able to take the simple sum of the arcs, but we have to add in an algebraic quantity, which is a multiple of the products of the sines of the three amplitudes.

The comparison of hyperbolic arcs hitherto has been matter of still greater complexity, as it has been usually handled simply by reducing each hyperbolic arc to two elliptic arcs. The complexity, of course, is thus doubled.

It seemed to me, however, that the ellipse is so completely the analogue of the hyperbola, and that it is so easy to pass from one to the other by an imaginary transformation, that a similar analogy ought to pervade the comparison of their arcs. Hence that there must exist some formula for the comparison of hyperbolic arcs, as simple as that for elliptic arcs.

A slight modification of Jacobi's second theorem has enabled me to find the analogue which I required.

If we take Euler's elliptic equation, and substitute the following changes,—

cosine of amplitude into *secant of amplitude*,
sine of amplitude into *tangent amplitude* $\times \sqrt{-1}$,
sine of modulus into *cosine of modulus*,

we leave the equation entirely unaltered, except in form. Even this form may be obtained directly by a simple transformation, of purely algebraic character. Hence it follows that all the consequences of this imaginary transformation are allowable.

If we apply these transformations to the elliptic integral of the first kind, we have Jacobi's second theorem.

If we apply them to the integral of the second kind, which represents the elliptic arc, we pass, after an obvious reduction, to the arc of the hyperbola. The algebraic addition to the sum of the arcs is simply changed from a product of sines to a product of tangents.

Other considerations enable us to verify the theorem, when once it is obtained. These verifications are merely algebraic, and would scarcely be intelligible if read aloud.

I have made use of my formulæ in the reduction of one class of the elliptic integral of the third kind for the purposes of tabulation.

In its ordinary form, this function has three variables, and, as Legendre justly remarks, a table of treble entry would be intolerable. If, therefore, it is to be tabulated at all, the first step in the question is to reduce it to a form involving two variables. By the help of some theorems of Jacobi, Legendre succeeded in effecting this where the elliptic function of the third kind has its parameter negative and less than the square of the modulus.

By the help of my formula, I have succeeded in reducing the case where the parameter is negative and greater than unity. The steps are the mere counterpart of Legendre's work in the second supplement of the treatise on Elliptic Functions.

Both Legendre's case and mine are of the logarithmic form, and can therefore be reduced to one another by algebraic transformation. The cases where the parameter is positive, or negative and intermediate between unity and the square of the modulus, are still unreduced. The difficulty is exactly analogous to that between the two cases of cubic equations, and this analogy is even carried into the very form of the solution.

Dr. Booth's application of the trigonometry of the parabola to the reducible case of the cubic equation, affords some hope that a correlative calculus may exist, particular cases of which may solve the cases now irreducible, just as the calculus of elliptic functions includes the trigonometry both of the parabola and the circle. My own investigations on this subject are still without any useful result.

Let

$$\cos \phi_1 = \cos \phi_2 \cos \phi_3 - \sin \phi_2 \sin \phi_3 \sqrt{1 - \sin^2 \theta \sin^2 \phi_1};$$

$$F\phi = \int \frac{d\phi}{(1 - \sin^2 \theta \sin^2 \phi)^{\frac{1}{2}}},$$

$$E\phi = \int (1 - \sin^2 \theta \sin^2 \phi)^{\frac{1}{2}} d\phi,$$

$$Y\phi = \int \frac{\cos^2 \theta d\phi}{(1 - \sin^2 \theta \sin^2 \phi)^{\frac{1}{2}} \cos^2 \phi d\phi},$$

- $$\left. \begin{aligned} (1) \quad & F\phi_1 - F\phi_2 + F\phi_3 = 0 \\ (2) \quad & E\phi_1 - E\phi_2 + E\phi_3 = \sin^2 \theta \sin \phi_1 \sin \phi_2 \sin \phi_3 \\ (3) \quad & Y\phi_1 - Y\phi_2 + Y\phi_3 = -\cos^2 \theta \tan \phi_1 \tan \phi_2 \tan \phi_3. \end{aligned} \right\} \text{Euler.}$$